EMITTANCE OF A TRANSPARENT PLASMA AND THE RELATION BETWEEN ELECTRON AND ION TEMPERATURES

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A revised expression is derived for the mean absorption coefficient of a hydrogen-type plasma, which in the limit of complete ionization becomes the standard formula for free-free absorption.

Consider the relation between the electron temperature T and the ion temperature T_i in an optically thin plasma. Radiative losses disturb the equilibrium between T and T_i; marked deviation of T/T_i from unity for constant T_i occurs for $T > 10^5$ °K for air and heavier gases.

In a radiation-cooled plasma, the difference of T/T_i from unity for the same T is less by about an order of magnitude than in the steady-state case. Exact data have been obtained for the dependence of T on T_i for an air plasma.

§1. General. Radiation losses cause deviation from thermodynamic equilibrium in an optically thin medium, which itself affects the radiative properties of the medium. This deviation in a small volume of given density has one of the following effects: 1) the populations u_l of the energy levels l cannot at all be described via equilibrium relations, or 2) these relations apply only to particular groups of u_l and not to all groups. In particular, the populations of the excited electron states (bound or free) are reduced either in an essentially nonequilibrium fashion or by reduction of T relative to T_i. The reduction in u_l in a radiating plasma with a given T has been discussed in detail [1, 2]. It has been shown for a thin hydrogen plasma [1] that $y \equiv u_l/u_l^*$ (in which u_l^* is the equilibrium value of u_l for T) for free electrons is much less than one when the density N_e of the free electrons is small. For instance, $y = 2 \cdot 10^{-5}$ at T = 5000° K (low degree of ionization) for a hydrogen plasma at normal pressure, while y = 0.84 at T = 15 000° K. The N_e giving

$$y = 1 \tag{1.1}$$

(local thermodynamic equilibrium in the electron states) is much higher for a gas with multiple ionization than for one with single ionization, because the cross-sections for collisional excitation and ionization by electrons decrease roughly as $(z+1)^{-3}$ (z = 0 for a neutral atom, z = 1 for a singly ionized atom, etc.) [2], while the probability of spontaneous emission increases as $(z+1)^4$ [3].

The free electrons in a transparent plasma are [2] in equilibrium with the populations of levels of principal quantum number n in hydrogen-type ions if

$$N_e \geqslant 7 \cdot 10^{18} \frac{(z+1)^7}{n^{17/2}} \left(\frac{kT}{I_z}\right)^{1/2} \text{cm}^{-3}, \tag{1.2}$$

in which I_z is the ionization potential of a hydrogen-type ion with charge $z - 1$.

Relation (1.1) is almost always obeyed for large n, but this is a fairly stringent condition for the lower excited levels. For example, (1.2) applies only for $n \geq 4$ for completely ionized air of normal density. However, there is considerable interest in a plasma optically thin relative to free-free and free-bound radiation but opaque for resonant radiation from the lower excited levels [3, 4], which do not satisfy (1.1) (for example, a low-temperature plasma is usually opaque to the Lyman series). If the emission from the first excited level is substantially reabsorbed, relation (1.1) is [2] obeyed in any case if

$$N_e \geqslant 10^{17} z^7 \left(\frac{kT}{I_z}\right)^{1/2} \left(\frac{E_2 - E_1}{I_z}\right)^3 \text{cm}^{-3}, \tag{1.3}$$

in which E_2 and E_1 are the energies of the levels with n = 2 and n = 1. If the plasma is also opaque for certain other resonant quanta, an inequality of the (1.3) type becomes less stringent [2]. Experiments with sparks [5] and shock waves [6] show that there is a zone where (1.1) is obeyed in a hot gas transparent for free-free and free-bound radiations. T may not equal T_i although (1.1) is obeyed, and this aspect is examined quantitatively here. We also examine the emissivity of a hot spatially homogeneous gas and the relation of T to T_i in a steady-state plasma (given T_i) and in a cooling plasma that satisfies (1.1) and is transparent for free-bound and free-free radiations.* It will be shown that, for air or a heavier gas,

$$(T_i - T) \ll T \tag{1.4}$$

is not obeyed for $T > 10^5$ °K.

§2. Steady state. Consider the relation of T to T_i for a radiating plasma in which T_i is kept constant in some way. T as a function of T_i is determined from the equality of the radiation flux to the energy received by the electrons from collisions with heavy particles, which is

$$3\, Nkz\, (T_i - T) \,/\, 2\,\tau \qquad (\tau = 252\, AT^{3/2}/z^2 N\Lambda), \tag{2.1}$$

in which N is the density of heavy particles, τ is the electron-ion relaxation time, A is the atomic weight of the ion and Λ is the Coulomb logarithm. If there is a mixture of ions, z means the mean ion charge, which equals the number of free electrons per heavy particle.

Expression (2.1) has been derived via Landau's relaxation equation, on the assumption that the relaxation does not cause the free-electron velocity to deviate from a Maxwellian distribution, which is justified, since the radiative cooling and collisional heating are slow relative to the establishment of the Maxwellian velocity distribution for electrons, which has a characteristic time $\tau_e = z\tau m/\mu$, in which m and μ are the masses of an electron and an ion. There would be an appreciable deviation from a Maxwellian distribution for the electrons of $T_i \gg T$, but this would require extremely high temperatures, whereas $T_i - T \ll T$ in all cases envisaged here.

The radiation emitted by unit volume of gas is [3]

$$4\sigma T^4 \varkappa\,(T), \tag{2.2}$$

in which $\varkappa$ is the mean absorption coefficient and σ is Stefan's constant.

The precise relation between T and T_i is dependent on the nature of the gas and will be considered for an air plasma after we have considered more approximately the relations for all gases.

We use the following approximate expression [3] for the absorption coefficient:**

$$\varkappa = 9 \cdot 10^{-24} \frac{N^2 (z-1) z^2}{T^{7/2}} \left(\frac{I_z}{kT}\right), \tag{2.3}$$

in which I_z is the ionization potential for ions with a mean charge $z - 1$.

We first need some revision of (2.3), which will be seen to be not very important for $I_z/kT > 1$, but which is necessary for the correct description of $\varkappa$ for a completely ionized plasma for $I_z/kT < 1$. Of course, in place of (2.2) and (2.3) we could use the data of [7], which

*Obedience to (1.1) for examination of the relation between T and T_i is required only insofar as it is necessary in order for N_e to be described by Saha's equation and for the main radiations from the plasma to be free-free and free-bound.

**The formula from [3] is here given in the revised form obtained by replacing the m of [3] by $z - 1$. We then use the method of [3], taking into account the fact that the maximum in $N_m \exp(-I_m/kT)$ does not coincide with the maximum N_m and lies at $m = z - 1$, which gives (2.3) for $z > 1$. This feature is important for the substitution of the correct value of I_z into (2.3).

were derived by direct calculation of the radiation losses of a hydrogen-type plasma via bremsstrahlung and recombination radiation. However, it is of interest, at least as regards method, to find the radiation losses of (2.2) by calculation of $\varkappa$.

§3. Mean absorption coefficient of a hydrogen-type plasma. Formula (2.3) is obtained by frequency averaging of the spectral coefficient $\varkappa_\nu$ for a hydrogen-type ion, with subsequent transformation by Saha's formula for an ion with the average charge [3]. The contribution from the bremsstrahlung absorption to $\varkappa$ is taken into account only by the Kramers-Unsold relation [3, 8]

$$\varkappa_\nu = \frac{aN_{z-1}z^2}{T^2x^3}\exp\left(x - \frac{I_z}{kT}\right),$$

$$h\nu < I_z, \quad a = 0.96 \cdot 10^{-7}\,\mathrm{cm}^2\,\mathrm{deg}^2, \quad x = \frac{h\nu}{kT}, \tag{3.1}$$

which applies for quanta with $h\nu \ll I_Z$. Moreover, (3.1) is applied throughout the range $h\nu < I_Z$ in deriving (2.3).

To convert correctly to $\varkappa_\nu$ for a completely ionized plasma we must allow for all free-free transitions and not use (3.1) at frequencies that do not obey $h\nu \ll I_Z$. The spectral coefficient for bound-free and free-free absorption is [3]

$$\varkappa_\nu = \frac{2az^2N_{z-1}}{T^2x^3}\left[\sum_{n=n^*}^{\infty}\frac{1}{n^3}e^{x_n - x_1} + \frac{1}{2x_1}e^{-x_1}\right]$$

$$\left(x_1 = \frac{I_z}{kT}, \quad x_n = \frac{x_1}{n^2}, \quad h\nu \geqslant \frac{x_1}{(n^*)^2}\right). \tag{3.2}$$

Formula (3.1) is derived from (3.2) by replacing the sum with respect to n by an integral. The contribution from n small must be taken into account separately in that approach. However, it is simpler to calculate $\varkappa$ by not using (3.1) at all and merely integrating (3.2) with respect to frequency [7] before summation with respect to n. Integration of (3.2) with respect to frequency (with respect to x with the weighting factor [3] $15\,x^3e^{-x}/\pi^4$) and summation over all types of ions gives

$$\varkappa = \frac{30a}{\pi^4T^2}\sum_{i=1}^{z_{\max}} N_{i-1}i^2\left(x_iS + \frac{1}{2}\right)e^{-x_i},$$

$$S = \sum_{n=1}^{\infty}\frac{1}{n^3} \approx 1.2, \tag{3.3}$$

which differs from the analogous expression in [3] by the factor 2/3 in front of the sum and by the additional 1/2 term in the parentheses. We get the following after combination of (3.3) with Saha's equation:

$$N_eN_i = 4.85\cdot 10^{15}\frac{g_i}{g_{i-1}}T^{3/2}e^{-x_i}N_{i-1},$$

and with $g_i/g_{i-1} \approx 1/2$ for the ratio of the statistical sums for hydrogen-type ions, and then averaging the sum with respect to i,

$$\varkappa \approx 6.2 \cdot 10^{-24}N^2z^3T^{-7/2}(2\,x_1S + 1). \tag{3.4}$$

This revised formula for $\varkappa$ is multiplied by $4\,\sigma T^4$ to give the corresponding expression in [7] if the latter is averaged with respect to ion charge. This result is entirely natural in view of Kirchoff's law and because the emittance in [7] was calculated on the basis of the same hydrogen-type approximation.

§4. Relation of T to T_i in a steady-state plasma. We equate (2.1) and (2.2), and use (3.4) with σ and $\varkappa$ to get

$$\theta \equiv \frac{T_i - T}{T} = \frac{8\sigma T^3\varkappa\tau}{3zNk} \approx 3.5\cdot 10^{-9}\frac{AT}{\Lambda}\left(\frac{SI_z}{kT} + \frac{1}{2}\right). \tag{4.1}$$

It follows from (4.1) that, for hydrogen, $\theta \approx 1$ only for $T \approx 10^{10}$ °K ($\Lambda \approx 10$ for a hydrogen plasma at such T); but the T for other elements are much less because $A \gg 1$, $I_Z/kT \gg 1$, and Λ is smaller. For instance, $\theta \approx 1$ for argon of normal density for T of $5 \cdot 10^5$ to 10^6 °K ($I_Z/kT \approx 10$, $\Lambda \approx 1$). These results are obtained via the approximate formula (3.4), whereas (1.4) may be violated at temperatures substantially less than this, because (3.4) for a hydrogen-type plasma gives $\varkappa$ as somewhat less than the more accurate formula (3.3), since in going from (3.3) to (3.4) no allowance is made in Saha's formula for the contribution of excited electronic levels to the electronic statistical sum for the ion.

The following are results for θ as a function of T for an air plasma of normal density, together with z and Λ, which have been calculated via data [9] for z and $\varkappa$ for air, the results for $\varkappa$ in [9] being from the integral Kramers-Unsold formula [8], which differs only slightly from (3.3) for $I_Z/kT > 1$:

$T\cdot 10^{-6}$ °K =	3.0	2.0	1.0	0.8	0.65	0.50	0.30	0.20	0.16	0.10	0.08
z =	7.2	7.2	6.8	6.2	5.6	5.2	5.0	4.5	3.9	3.1	2.4
Λ =	3.2	2.6	1.7	1.7	1.6	1.5	0.8	0.53	0.64	0.59	1.0
$\varkappa$(cm^{-1}) =	$6.1\cdot 10^{-3}$	$1.9\cdot 10^{-2}$	0.16	0.32	0.87	2.7	12	26	28	45	54
θ =	0.98	0.62	0.41	0.42	0.61	0.82	0.67	0.53	0.27	0.11	0.06

This shows that the difference between T and T_i becomes appreciable for $T = 10^5$ °K and that $\theta \approx 0.5$ for $T = 2 \cdot 10^5$ °K, with little change when T increases up to $2 \cdot 10^6$ °K. Equation (4.1) describes closely the subsequent increase in θ with T. The dependence of θ on the gas density has been included in Λ. The difference between T and T_i increases with the density on account of the fall in Λ.

§5. Cooling plasma. The time dependence of T and T_i here is found by solving a system of two ordinary differential equations, but an approximate solution of good accuracy can be obtained in another way, because the specific heat of the electron gas (which includes the specific heat of ionization) is much greater than the specific heat of the gas of heavy particles, so the electron gas cools fairly slowly, and T_i can follow T. As a first approximation we put $dT/dt = dT_i/dt$ to get

$$dT_i\,/\,dt = 4\,\sigma T^4\varkappa\,/\,C_v, \tag{5.1}$$

in which t is time and C_V is the equilibrium value of the specific heat at constant volume for a plasma of temperature T.

We substitute (5.1) into the relaxation equation for T_i,

$$dT_i\,/\,dt = z\,(T - T_i)\,/\,\tau,$$

to get

$$\theta = 4\,\sigma T^3\varkappa\tau\,/\,(zC_v). \tag{5.2}$$

Comparison of (5.2) with (4.1) shows that θ for a cooling plasma is less than that for the same T in the steady state by a factor $2C_V/3Nk$. This ratio is z + 1 for a completely ionized plasma, while for incomplete ionization it is greater than z + 1 on account of the contributions to C_V from the specific heat of ionization and the excitation of the electronic levels of the bound states.

If $T = T_i = T_0$ initially, the condition of (5.2) is reached in a time Δt, during which the plasma cools to

$$T \approx T_0 - 4\,\sigma T_0^4\varkappa\tau\,/\,(zC_v), \qquad T_i \approx T_0$$

It follows from (5.1) and (5.2) that $\Delta t \approx \tau/z$.

The following are values for $2C_V/3Nk$ calculated via [9] and also θ for a cooling air plasma of normal density for the same T as for the steady state:

$T\cdot 10^{-6}$ (°K) =	3	2	1.0	0.8	0.65	0.5	0.3	0.2	0.16	0.1	0.084
$2c_v/3Nk$ =	7.8	7.9	21	28	24	11	8.1	14	17	15	14
θ =	0.13	0.087	0.02	0.015	0.025	0.074	0.083	0.038	0.016	0.007	0.004

This shows that θ is much less than in the corresponding steady states and does not exceed 9% for all $T < 2 \cdot 10^6$ °K.

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